J. Fluid Mech. (1972), *vol.* 56, *part* 1, *pp.* 101–119
Printed in Great Britain

On the radiation and scattering of short surface waves. Part 1

By F. G. LEPPINGTON
Department of Mathematics, Imperial College, London

(Received 2 May 1972)

The radiation and scattering of time-periodic surface waves by partially immersed objects is investigated in the short-wave asymptotic limit $\epsilon \to 0$, where ϵ is a non-dimensional wavelength. Details are given for the prototype radiation and scattering problems in which the fluid has infinite depth and the body is a two-dimensional dock of finite width and zero thickness. The solutions are then generalized to deal with other two-dimensional geometries, with the restriction that the ends of the obstacle are horizontal for a distance of many wavelengths. The method of matched expansions is used. A first approximation ϕ_0, presumed to be a good estimate for the potential throughout most of the fluid region, is obtained by replacing the free-surface condition by its formal limit $\phi_0 = 0$. In the vicinity of the ends of the obstacle, the correct surface condition is used but the geometry of the problem is simplified. The remaining surface layers are dealt with by superimposing on the function ϕ_0 regular wave trains of the appropriate amplitude.

1. Introduction

If a body is partially immersed in a fluid and undergoes simple harmonic oscillations, then its motion will ultimately produce a system of regular sinusoidal wave trains, radiating outwards, that are confined within a thin layer close to the free surface. Similarly, if the body is held fixed and is irradiated by a regular wave train, then it will scatter surface waves that travel outwards. The purpose of this work is to predict the amplitude of these induced wave trains, in the limit of wavelengths very small compared with a characteristic length scale a of the obstacle. For simplicity, the radiating and scattering bodies are taken to be two-dimensional, with generators parallel to the z axis and fluid in the half-space $y > 0$ of a Cartesian co-ordinate system.

Since we are considering motions that are time-periodic, the velocity potential is of the form $\mathscr{R}\{\phi(x,y)e^{-i\omega t}\}$, where ω is the radian frequency; the time dependence is described by the factor $e^{-i\omega t}$, which appears throughout and will henceforth be suppressed.

A typical radiation problem, in which the normal velocity on the wave maker S has the prescribed value $\mathscr{R}\{V(\mathbf{x})e^{-i\omega t}\}$, involves finding a harmonic function $\phi(x,y)$ that satisfies the linearized boundary conditions

$$\partial\phi/\partial n = V \quad \text{on} \quad S \tag{1.1}$$

and
$$\phi + \epsilon\phi_y = 0, \quad y = 0 \tag{1.2}$$
on the mean free surface, where $\epsilon = g/\omega^2$ is $(1/2\pi)$ times the wavelength of surface waves, g is the acceleration due to gravity and ϕ_y is used to denote $\partial\phi/\partial y$. In addition we require an edge condition
$$\delta(\partial\phi/\partial\delta) \to 0 \quad \text{as} \quad \delta \to 0, \tag{1.3}$$
where δ is the distance from a point where the obstacle meets the fluid, and a condition to ensure that waves travel outwards at infinity. Thus
$$\phi \sim A_{\pm}\exp\{(\pm ix - y)/\epsilon\} \quad \text{as} \quad x \to \pm\infty; \tag{1.4}$$
the asymptotic evaluation of the constants A_+ and A_-, for small ϵ, is the main objective of this work.

For the related scattering problem, in which a wave train ϕ_i is incident upon a fixed body, the function V of (1.1) is set equal to zero, and the radiation condition (1.4) is applied to the scattered potential $\phi - \phi_i$.

It is important to note the fact that the outward wave trains (1.4) are formed at a distance of many *wavelengths* from the wave maker S. Taking x to be positive, for example, and using (x_0, y_0) to denote the point on S with the largest x co-ordinate, it is asserted that, apart from wave-free terms that vanish as $x - x_0 \to \infty$,
$$\phi \sim A_+\exp\{(ix - y)/\epsilon\} \quad \text{as} \quad (x - x_0)/\epsilon \to \infty. \tag{1.5}$$
This means that a regular wave train is formed when the distance $x - x_0$ from S is much greater than the wavelength, with $x - x_0$ either small or large compared with the size of S.

To establish this property, the potential $\phi(x, y)$ at any point is represented in terms of its values $\phi(x', y')$ on S, together with the Green's function $G(x', y'; x, y)$ specified by the conditions
$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right) G = \delta(x' - x)\,\delta(y' - y), \quad y > 0, \quad y' > 0, \tag{1.6}$$
with
$$G + \epsilon\,\partial G/\partial y' = 0 \quad \text{when} \quad y' = 0, \tag{1.7}$$
and a radiation condition at infinity. An application of Green's theorem to the functions $\phi(x', y')$ and G leads to the identity
$$\phi(x, y) = \int_S \left\{G V(x', y') - \phi(x', y')\frac{\partial G}{\partial n'}\right\} dx'dy', \tag{1.8}$$
where n' denotes the normal from S into the fluid.

Now the function G is readily calculated and is found (cf. Holford 1964) to have, apart from wave-free terms, the far-field form
$$G \sim -i\exp\{i(x - x')/\epsilon - (y + y')/\epsilon\} \quad \text{as} \quad (x - x')/\epsilon \to \infty. \tag{1.9}$$
This approximation is uniformly valid for all (x', y') on S, provided $(x - x_0) \gg \epsilon$, so that substitution of (1.9) into formula (1.8) yields the desired result (1.5); the amplitude constant A_+ is seen to be given in terms of the (unknown) potential on S as
$$A_+ = -i\int_S \left\{V(x', y') - \phi(x', y')\frac{\partial}{\partial n'}\right\} \exp\left(\frac{-ix' - y'}{\epsilon}\right) dx'\,dy', \tag{1.10}$$
though this will not be used here to calculate A_+.

A similar argument holds for finite depth h, using the Green's function whose normal derivative vanishes at $y = h$. The wave trains (1.4) and (1.5) have ϵ replaced by ϵ_1, given by $\epsilon = \epsilon_1 \coth(h/\epsilon_1)$, and there are some additional wave-free terms that vanish when $x - x_0 \gg h$.

An asymptotic solution of the problem (1.1)–(1.4) is sought by posing different approximations in different regions. Since ϵ is supposed small it is natural to take, as a first approximation, a function ϕ_0 that is obtained by formally setting $\epsilon = 0$ in the boundary condition (1.2). Such an approximation could reasonably be expected to be valid throughout most of the fluid region, but it is obviously incorrect near the free surface, since ϕ_0 does not have the required wavelike behaviour there. In view of the fact that regular wave trains decay rapidly with increasing y/ϵ it seems plausible to suppose that this 'outer' approximation holds good up to distances that are many wavelengths $2\pi\epsilon$ from the free surface: this distance is small on the length scale a of the obstacle, since ϵ/a is small.

At points that are very close (on the length scale a) to an end of the body, where it meets the free surface, an observer will be well aware of the waviness of the surface, but will be aware of only the *local* shape of the obstacle. Thus the approximations near the two ends involve simplifications in the geometry of the body, with solutions that are slowly varying functions of independent variables scaled with respect to *wavelength*; the formal transformations used are described in detail below.

Since these 'inner solutions' are valid only well within a distance a from the ends, there is some difficulty in assigning boundary conditions at infinity: points at great distances from the ends lie outside the region of validity of the inner approximations. A similar difficulty arises in deciding on the correct edge conditions for the outer approximation ϕ_0, which is not valid at the ends. The idea of matched expansions provides the means of completing the specifications for the inner and outer solutions. For if δ denotes the distance from one of the ends of the obstacle, then the outer estimate is presumed valid for $\delta \gg \epsilon$ and the inner solution is valid for $\delta \ll a$; if $\epsilon \ll \delta \ll a$ there is evidently a common region of validity, in which both approximations must be equivalent in some sense. The formal machinery for exploiting this equivalence has been propounded by Van Dyke (1964), and a description of the precise matching principle used here is given in the appendix.

Detailed analyses are given, in §§2 and 4, for the prototype radiation and scattering problems in which the geometry is that of a finite dock on a fluid of infinite depth.

Asymptotic solutions to these problems have already been given by Holford (1964) and Leppington (1968, 1970). Holford has shown that they can be formulated as integral equations, of the second kind, with kernels that are sufficiently small to ensure solution by iteration in the short-wave limit. This method has the advantage of providing a rigorous derivation of the leading term for the amplitude of the radiated and scattered waves; it is not useful for calculating higher order terms, nor is it readily capable of extension to different geometries.

The present approach is not rigorous, but is amenable to generalization to other geometries and able to deal with higher order expansions. Section 2 includes the analysis for higher order approximations in the radiation problem for the finite dock with fluid of infinite depth. Extensions to other geometries are described in the remaining sections with explicit results given in §§3 and 5 for the radiation and scattering by a finite dock on a fluid of finite depth; scattering by a T-shaped dock, with fluid of infinite depth, is described in §6, where further generalizations are also suggested.

2. Prototype radiation problem: the finite dock

A detailed description will be given here of the prototype problem of waves generated by the vertical motion of a dock of negligible thickness and width $2a$ on the surface of a fluid of infinite depth. Co-ordinates are chosen so that the half-width a of the dock is the unit of length, with fluid in the half-space $y > 0$. The dock occupies the region $y = 0$, $|x| < 1$, and undergoes small amplitude vibrations of velocity $\mathscr{R}\{V(x)e^{-i\omega t}\}$, producing a two-dimensional irrotational velocity field of potential $\mathscr{R}\{\phi(x,y)e^{-i\omega t}\}$. The time factor $e^{-i\omega t}$ will be suppressed, and the potential $\phi(x,y)$ therefore satisfies the conditions

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = 0 \quad (y > 0), \tag{2.1}$$

$$\partial\phi/\partial y = V(x) \quad (y = 0, |x| < 1), \tag{2.2}$$

$$\phi + \epsilon\,\partial\phi/\partial y = 0 \quad (y = 0, |x| > 1). \tag{2.3}$$

Here $\epsilon = g/\omega^2 a = g/\omega^2$ is $(1/2\pi)$ times the ratio of wavelength to dock half-width, and is taken to be small. Uniqueness requires further *edge conditions*,

$$\delta(\partial\phi/\partial\delta) \to 0 \quad \text{as} \quad \delta^2 = (x\pm 1)^2 + y^2 \to 0, \tag{2.4}$$

and a *radiation condition*,

$$\phi \sim A_\pm \exp\{(\pm ix - y)/\epsilon\} \quad \text{as} \quad x \to \pm\infty, \tag{2.5}$$

together with terms that vanish as $x^2+y^2 \to \infty$. The condition (2.5) ensures that the surface waves travel *outwards* at large distance from the dock, and the asymptotic evaluation of the constants $A_\pm$ is the primary objective of this investigation. It is sufficient to calculate A_+, since the symmetry of the problem may then be used to obtain A_- from A_+ by simply changing $V(x)$ to $V(-x)$.

(i) *The outer approximation*

A first approximation to the potential, and one that is presumed to be valid at all points that are many wavelengths distant from the free surface, is specified by formally taking the limit $\epsilon \to 0$ in (2.3), to get

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi_0 = 0 \quad (y > 0),\\ \text{with}\quad &\partial\phi_0(x,0)/\partial y = V(x) \quad (|x| < 1); \quad \phi_0(x,0) = 0 \quad (|x| > 1). \end{aligned}\right\} \tag{2.6}$$

We may no longer insist on the condition (2.4) at the two edges, which lie outside the region of validity of ϕ_0; nevertheless, it transpires that ϕ_0 does in fact

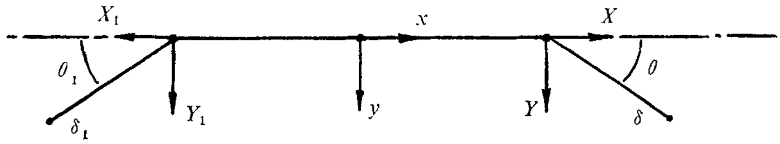

FIGURE 1. Co-ordinate system. The local co-ordinates (X, Y) and (δ, θ) are centred on the edge $x = 1$, and (X_1, Y_1) and (δ_1, θ_1) are centred on $x = -1$.

satisfy such edge conditions, as will be indicated later. The specifications (2.6) permit no surface waves, of course, so that the condition at infinity is simply that ϕ_0 should vanish at great distances from the origin.

The problem (2.6) is a simple one that can be solved by conformal transformation. In particular, Holford (1964) shows that the solution satisfying an edge condition is such that

$$\frac{\partial\phi_0(x,0)}{\partial y} = -\frac{1}{\pi}(x^2-1)^{-\frac{1}{2}}\int_{-1}^{1}\frac{(1-x'^2)^{\frac{1}{2}}}{(x-x')}V(x')\,dx' \quad (|x| > 1). \tag{2.7}$$

To this particular solution for ϕ_0 may be added any eigensolutions that vanish on the free surface and have zero normal derivative on the dock. Such solutions ϕ_E have singularities, at one or both of the edges, where $\phi_E = O(\delta_\pm^{-n+\frac{1}{2}})$,

$$\delta_\pm^2 = (x\pm 1)^2 + y^2,$$

with n a positive integer. It will be shown later that such eigensolutions added to ϕ_0 must have zero coefficients.

In order to calculate the behaviour of ϕ_0 near the edge $x = +1$, $y = 0$, we set $x = 1+\delta$ and expand the integral (2.7) for small δ to get

$$\frac{\partial\phi_0(1+\delta,0)}{\partial y} \sim -\frac{1}{\pi}(2\delta)^{-\frac{1}{2}}\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}V(x)\,dx. \tag{2.8}$$

An approximation that is valid for all points in the vicinity of the edge follows at once from (2.8) together with (2.6). For if (δ, θ) are local polar co-ordinates based on the edge, given by $x = 1+\delta\cos\theta$, $y = \delta\sin\theta$ (see figure 1), then the solution of Laplace's equation that is consistent with (2.8), and $\phi_0(x, 0) = 0$, $x > 1$, is given by

$$\phi_0(\delta,\theta) \sim I\delta^{\frac{1}{2}}\sin\tfrac{1}{2}\theta, \quad \delta\to 0, \tag{2.9}$$

where

$$I = -\frac{2^{\frac{1}{2}}}{\pi}\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}V(x)\,dx. \tag{2.10}$$

The corresponding estimate near the other edge, in terms of the variables (δ_1, θ_1) defined by $x = -1-\delta_1\cos\theta_1$ and $y = -\delta_1\sin\theta_1$ (see figure 1), is obtained from (2.9) and (2.10) on replacing $V(x)$, δ and θ by $V(-x)$, δ_1 and θ_1.

(ii) *The inner approximations*

At points that are very close to an edge (i.e. at distances much less than the dock width), an observer will be well aware of the waviness of the free surface, but will become unaware of the presence of the other edge in the limit $\epsilon\to 0$. Evidently the significant length scale is that of wavelength, suggesting the co-ordinate rescaling

$$x = 1+\epsilon X, \quad y = \epsilon Y, \tag{2.11}$$

to examine the structure of the solution near the edge $x = \pm 1$, with

$$\phi(x,y) = \phi(1+\epsilon X, \epsilon Y) = \Phi(X, Y) \tag{2.12}$$

expressing the potential as a function X and Y. For the potential near the other edge, we write $x = -1-\epsilon X_1$ and $y = \epsilon Y_1$, with $\phi(x,y) = \Psi(X_1, Y_1)$. In terms of the co-ordinates (2.11) and (2.12) the boundary-value problem for the potential takes the form

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\Phi = 0 \quad (Y > 0), \tag{2.13}$$

$$\partial\Phi/\partial Y = \epsilon V(1+\epsilon X) \quad (Y = 0, \quad -2/\epsilon < X < 0), \tag{2.14}$$

$$\Phi + \partial\Phi/\partial Y = 0 \quad (Y = 0, \quad X > 0 \quad (\text{or} \quad X < -2/\epsilon)). \tag{2.15}$$

It is required to find an approximation that is valid at points where the distance from the edge is small compared with the dock width, whence

$$R = (X^2+Y^2)^{\frac{1}{2}} \ll 1/\epsilon.$$

It is therefore natural to expand the function $V(1+\epsilon X)$ as a Taylor series and to apply the boundary condition (2.14) for all negative X. As for the boundary condition for large values of $R = \delta/\epsilon$, this must be chosen so that the two approximations ϕ_0 and Φ match smoothly. For if δ is chosen so that

$$\epsilon \ll \delta \ll 1, \tag{2.16}$$

then both approximations are valid. That is to say, an observer at distance δ from the edge is *close to the edge* in the sense that he is a fraction of dock width away, with the estimate (2.9) being valid; also, our observer is at *great distance from the edge* on a wavelength scale, whence the inner variable R is large. Thus we require

$$\underset{\delta\to 0}{\text{asymptotic limit}}\ \phi(\delta,\theta) = \underset{R\to\infty}{\text{asymptotic limit}}\ \Phi(R,\theta), \tag{2.17}$$

this being a special case of the more general matching principle, due to Van Dyke (1964), that is discussed briefly in the appendix. Similar considerations near the other edge require that $\lim \phi(\delta_1 \to 0) = \lim \Psi(R_1 \to \infty)$. This equivalence of the two approximations provides the missing boundary condition at infinity for our inner approximation Φ; the formal procedure is to write our outer estimate ϕ_0 in terms of the inner variables X and Y and expand for small ϵ to get, using (2.9),

$$\lim_{\delta\to 0} \phi_0 = \epsilon^{\frac{1}{2}} I R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta, \quad \epsilon \ll \delta \ll 1,$$

suggesting that

$$\Phi \sim \epsilon^{\frac{1}{2}} \Phi_0, \tag{2.18}$$

with the behaviour of Φ_0 at large distance chosen to satisfy the matching requirements described above. Thus our problem for Φ_0 is

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}\right)\Phi_0 = 0 \quad (Y > 0),\\ &\partial\Phi_0/\partial Y = 0 \quad (Y = 0, \quad X < 0),\\ &\Phi_0 + \partial\Phi_0/\partial Y = 0 \quad (Y = 0, \quad X > 0),\end{aligned}\right\} \tag{2.19}$$

together with an edge condition at $R = 0$ and the condition at infinity that

$$\Phi_0 \sim I R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta + \text{outgoing wave train} \quad \text{as} \quad R \to \infty. \tag{2.20}$$

The necessity for an *outgoing wave condition* on the inner potential Φ_0 follows immediately from the far-field estimate (1.5); this ensures the onset of a regular outgoing wave train at distances $x-1 \gg \epsilon$ (i.e. $X \gg 1$), even within the inner region $x-1 \ll 1$.

It is convenient to subtract out the large wave-free potential from Φ_0 by writing

$$\Phi_0 = I(R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta - \tfrac{1}{2}G_0), \tag{2.21}$$

whence G_0 is the harmonic function that has the boundary conditions

$$\partial G_0/\partial Y = 0 \quad (Y = 0, X < 0), \quad G_0 + \partial G_0/\partial Y = X^{-\frac{1}{2}} \quad (Y = 0, X > 0),$$

with an edge condition at $R = 0$ and outgoing wave condition as $X \to \infty$. This potential is that due to a pressure distribution proportional to $P_0(X) = X^{-\frac{1}{2}}$ over the surface $X > 0$, with a rigid dock over the rest of the surface, $X < 0$, and the solution for problems of this type is given by Holford (1964). If $P_0(X)$ is written in the form of a Laplace transform

$$P_0(X) = \int_0^\infty \Pi_0(s) e^{-sX} ds, \tag{2.22}$$

then

$$G_0(X, Y) = \int_0^\infty \Pi_0(s)\, \Theta(X, Y; s)\, ds, \tag{2.23}$$

where

$$\left.\begin{aligned} \Theta(X, Y; s) &= -i2^{\frac{1}{2}} \frac{\Lambda(s)}{1+s^2} \exp\left[i(X - \tfrac{1}{8}\pi) - Y\right] + \frac{(\cos sY + s \sin sY)}{1+s^2} e^{-sX} \\ &\quad + \frac{1}{\pi} \frac{\Lambda(s)}{s-i} ⨍_0^\infty \frac{t \cos tY - \sin tY}{(t+i)\Lambda(t)} e^{-tX} \frac{dt}{t-s} \quad (X > 0), \\ \Theta(X, Y; s) &= -\frac{1}{\pi} \frac{\Lambda(s)}{s-i} \int_0^\infty \frac{\Lambda(t) \cos tY e^{tX}}{t(t-i)} \frac{dt}{t+s} \quad (X < 0). \end{aligned}\right\} \tag{2.24}$$

The symbol $⨍_0^\infty$ denotes a Cauchy principal value, and

$$\Lambda(s) = s^{\frac{1}{2}}(1+s^2)^{\frac{1}{4}} \exp\left\{-\frac{1}{\pi}\int_0^s \frac{\log u}{1+u^2} du\right\} \tag{2.25}$$

is analytic except for a branch cut from 0 to $-\infty$, with

$$\Lambda(te^{\pm i\pi}) = -t(t \mp i)/\Lambda(t) \quad (t \text{ real}). \tag{2.26}$$

In the present case the pressure $P_0(X) = X^{-\frac{1}{2}}$ (whence $\Pi_0(s) = (\pi s)^{-\frac{1}{2}}$) and is seen from the first term of (2.24) to produce a surface wave train given by

$$\epsilon^{\frac{1}{2}}\Phi_0 \sim A_0 e^{i/\epsilon} e^{iX-Y} = A_0 \exp\{(ix-y)/\epsilon\} \quad \text{as} \quad X \to \infty, \tag{2.27}$$

where the amplitude constant is given by

$$A_0 = iI\epsilon^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{8}i\pi - i/\epsilon} \int_0^\infty \frac{\Lambda(s)\, ds}{s^{\frac{1}{2}}(1+s^2)} = iI\epsilon^{\frac{1}{2}}(\tfrac{1}{2}\pi)^{\frac{1}{2}} e^{-i\frac{1}{8}\pi - i/\epsilon}. \tag{2.28}$$

The integral of (2.28) is found to have the value π by integrating the function $1/s^{\frac{1}{2}}\Lambda(s)$ round a closed contour, in the complex-s plane, slotted along the negative real axis and bounded by circles of radii ρ_1 and ρ_2, finally letting $\rho_1 \to 0$ and $\rho_2 \to \infty$.

(Similar integrals are treated in detail by Holford 1964; Leppington 1970.) The solution (2.21)–(2.24) for $\Phi_0(X,Y)$ has an obvious analogue $\Psi_0(X_1,Y_1)$, describing the potential near the other edge $x=1$; we simply replace (X, Y) by (X_1, Y_1), and I by I', which is calculated by changing $V(x)$ to $V(-x)$ in the integral (2.10).

It is possible to justify, at this stage, an earlier assertion that no eigenfunction ϕ_E can be added to the outer approximation ϕ_0. For the addition of such a term would produce a term like $C\delta^{-n+\frac{1}{2}}\sin\frac{1}{2}\theta$ at one or other of the edges. Taking $n=1$, for example, this would require an inner expansion of the form $\Phi\sim\epsilon^{-\frac{1}{2}}\Phi_0$, with Φ_0 specified by (2.19), together with the condition $\Phi_0\sim CR^{-\frac{1}{2}}\sin\frac{1}{2}\theta$ at infinity, and no such function exists other than the trivial one $\Phi_0=0$, with $C=0$.

(iii) *The surface wave region*

The function $\epsilon^{\frac{1}{2}}\Phi_0$ gives an estimate for ϕ in the region that is close, on a dock-width scale, to the edge $x=1$; at the outer extremity of this near-field region, where X is large and $x-1$ is small, it has been seen that a regular wave train has been formed. Our description of the potential field requires knowledge of the solution in the important remaining region that is close to the free surface, and many wavelengths from the edge.

Now the far-field estimate (1.5) shows that the aforementioned wave train, formed towards the outer extremity of the inner region at distance much greater than a wavelength from the edge, propagates without change along the rest of the free surface. Thus for the remaining surface wave region, we simply continue the outer approximation ϕ_0 up to the surface, and add the regular wave train (2.27) that has been formed near the edge. That is,

$$\phi\sim\phi_0(\mathbf{x})+A_0\exp\{(ix-y)/\epsilon\}\quad(x\gg\epsilon),\tag{2.29}$$

in the surface wave region. This function obviously matches with ϕ_0, since the additional wave term is exponentially small as y increases; the matching with the inner field $\epsilon^{\frac{1}{2}}\Phi_0$ is ensured by our very construction of Φ_0. To this order, then, the regular wave train propagating towards $x=\infty$ has the amplitude constant A_0 given by (2.28), which is in agreement with earlier results (Holford 1964; Leppington 1970). The corresponding amplitude A_- is obtained from (2.28) by changing I to I' (i.e. $V(x)$ to $V(-x)$).

(iv) *Higher order approximations*

The way is now clear to calculate higher order approximations, by continuing the outer and inner expansions, ϕ, and Φ and Ψ, to the required accuracy, matching at each of the edges. Finally the surface-layer estimate is written down by superimposing ϕ on the regular wave trains arising from Φ and Ψ. To illustrate the method, details are given below for the special case of constant velocity; the same procedure holds for general $V(x)$, and involves only the additional problem of calculating the expansions analogous to these (2.38) that are given here for constant V. Since the problem is linear, we may take the constant velocity to have the value $V=1$, without any loss of generality.

In order to investigate the development of the outer expansion $\phi(\mathbf{x};\epsilon)$, it is necessary to expand the edge solutions Φ_0 and Ψ_0 a little further. It is found from the exact solution (2.21)–(2.24), with $\Pi_0(s) = \pi^{-\frac{1}{2}}s^{-\frac{1}{2}}$ and $I = -2^{\frac{1}{2}}$, that, for large R,

$$\Phi_0 \sim -2^{\frac{1}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta + \frac{1}{2^{\frac{1}{2}}\pi}\{(\pi-\theta)\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta\log R\}R^{-\frac{1}{2}} + \frac{1}{2^{\frac{1}{2}}\pi}\{\log 4+\gamma+1-i\pi\}R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta, \quad (2.30)$$

together with a regular wave train, where $\gamma = 0{\cdot}5772\ldots$ is the Euler constant. The symmetry of the problem ensures an identical expression for $\Psi_0(R_1, \theta_1)$ on replacing R by R_1. Rewriting the estimate (2.30) in terms of the outer variable $\delta = \epsilon R$, it is seen that $\epsilon^{\frac{1}{2}}\Phi_0(\delta/\epsilon, \theta)$ contains terms of order $\epsilon\log\epsilon$ and ϵ; this suggests that the outer approximation ϕ has the form

$$\phi \sim \phi^{(1)} = \phi_0 + \epsilon\log\epsilon\phi_1 + \epsilon\phi_2 \quad \text{as} \quad \epsilon \to 0,\ \mathbf{x}\ \text{fixed}, \quad (2.31)$$

in which the superscript (1) denotes an expansion up to and including terms of order ϵ^1. On substituting (2.31) into the boundary-value problem (2.1)–(2.3) and formally equating like terms in ϵ, the functions ϕ_0, ϕ_1 and ϕ_2 are seen to satisfy the Laplace equation, together with the conditions

$$\partial\phi_0(0,y)/\partial y = 1, \quad |x| < 1; \quad \phi_0(x,0) = 0, \quad |x| > 1; \quad (2.32)$$

$$\partial\phi_1(x,0)/\partial y = 0, \quad |x| < 1; \quad \phi_1(x,0) = 0, \quad |x| > 1; \quad (2.33)$$

$$\partial\phi_2(x,0)/\partial y = 0, \quad |x| < 1; \quad \phi_2(x,\theta) = -\partial\phi_0/\partial y, \quad |x| > 1, \quad (2.34)$$

and each of these functions must also vanish at infinity.

These are quite simple problems that can be solved by conformal-transformation methods. It is found that the solutions are given, in terms of $z = x+iy$, by

$$\phi_0 = \mathscr{R}\{i(z^2-1)^{\frac{1}{2}} - iz\}, \quad (2.35)$$

$$\phi_1 = A\mathscr{R}\left\{i\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} - i\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right\}, \quad (2.36)$$

and

$$\phi_2 = \mathscr{R}\frac{i}{\pi}\left\{\frac{z}{(z^2-1)^{\frac{1}{2}}}\left[\log\frac{z-1}{z+1} - i\pi\right] + \frac{2}{(z^2-1)^{\frac{1}{2}}} + i\pi\right\} + B\mathscr{R}\left\{i\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} - i\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right\}. \quad (2.37)$$

The square root and logarithmic functions have branch cuts from $z = \pm 1$ that lie in the half-plane $y < 0$, and are chosen to be real and positive when z is a large real number. The constants A and B that occur in the eigensolutions ϕ_1 and ϕ_2 are arbitrary real numbers, to be determined by matching; the equal weighting given to each of the separate eigensolutions $i(z\pm 1)^{\frac{1}{2}}/(z\mp 1)^{\frac{1}{2}}$ has been chosen on account of the symmetry of the problem and would not be appropriate if V were not even in x. Further eigensolutions $\mathscr{R}\{i(z\pm 1)^{n+\frac{1}{2}}/(z\mp 1)^{n+\frac{1}{2}}\}$ could be added to any of the functions ϕ_0, ϕ_1 or ϕ_2, but are rejected by an argument similar to that discussed earlier: such additional functions would have higher order singularities at an edge and could not be matched to any inner expansion.

The matching procedure near the edge $x = 1$ requires that we write the outer expansion $\phi^{(1)}$, given by (2.31), in terms of the inner variables by means of the transformation $z = 1+\epsilon Re^{i\theta}$ and expand formally for small ϵ, up to terms of convenient order ϵ^m. Choosing $m = \frac{3}{2}$, the resulting function is denoted by the symbol $\phi^{(1,\frac{3}{2})}$. It is found that

$$\begin{aligned}\phi^{(1,\frac{3}{2})} &= \epsilon^{\frac{1}{2}}\log\epsilon\{(2^{\frac{1}{2}}A+1/2^{\frac{1}{2}}\pi)R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta\}\\ &\quad+\epsilon^{\frac{1}{2}}\left\{-2^{\frac{1}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+\frac{1}{2^{\frac{1}{2}}\pi}(\sin\tfrac{1}{2}\theta\log R+(\pi-\theta)\cos\tfrac{1}{2}\theta)R^{-\frac{1}{2}}\right.\\ &\quad\left.-\frac{1}{2^{\frac{3}{2}}\pi}(\log 4-4-4B)R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta\right\}+\epsilon\{R\sin\theta-1\}\\ &\quad+\epsilon^{\frac{3}{2}}\log\epsilon\{R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta(A2^{-\frac{3}{2}}-3/2^{\frac{5}{2}}\pi)\}+\epsilon^{\frac{3}{2}}\left\{-2^{-\frac{1}{2}}R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta\right.\\ &\quad\left.+(3/2^{\frac{5}{2}}\pi)\left((\pi-\theta)\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta\log R\right)R^{\frac{1}{2}}+\frac{1}{2^{\frac{3}{2}}\pi}(\tfrac{3}{2}\log 2+2+B)R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\right\},\end{aligned} \tag{2.38}$$

and this dictates the form of the inner expansion $\Phi(R,\theta)$. Since the leading term for Φ is already known to be of order $\epsilon^{\frac{1}{2}}$ (cf. equation (2.19)), the first term in formula (2.38) must vanish, whence

$$A = -1/2\pi. \tag{2.39}$$

The expression (2.38) implies further that Φ has an expansion, up to order $\epsilon^{\frac{3}{2}}$, of the form

$$\Phi \sim \Phi^{(\frac{3}{2})} = \epsilon^{\frac{1}{2}}\Phi_0+\epsilon\Phi_1+\epsilon^{\frac{3}{2}}\log\epsilon\,\Phi_2+\epsilon^{\frac{3}{2}}\Phi_3. \tag{2.40}$$

On substituting this expression into the governing equations (2.13)–(2.15), with $V = 1$, and finally taking the limit $\epsilon\to 0$, it is found that each of the constituent potentials Φ_0, Φ_1, Φ_2 and Φ_3 is harmonic and satisfies an edge condition, and the free-surface condition $\Phi_i+\partial\Phi_i/\partial Y = 0$, $X > 0$, $Y = 0$. The function Φ_1 has derivative $\partial\Phi_1/\partial Y = 1$ on the semi-infinite dock $X < 0$, $Y = 0$; the other potentials have zero normal derivative on the dock. Finally, boundary conditions at infinity are provided by the matching condition $\Phi^{(\frac{3}{2},1)} = \phi^{(1,\frac{3}{2})}$, which means that the asymptotic limits of the potentials Φ_0, Φ_1, Φ_2 and Φ_3, for large R, are given by the appropriate functions that occur in the curly brackets of formula (2.38). Thus we have, for example,

$$\begin{aligned}\Phi_0 \sim -2^{\frac{1}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta+\frac{1}{2^{\frac{1}{2}}\pi}(\sin\tfrac{1}{2}\theta\log R+(\pi-\theta)\cos\tfrac{1}{2}\theta)R^{-\frac{1}{2}}\\ -\frac{1}{2^{\frac{3}{2}}\pi}(\log 4-4-4B)R^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta,\end{aligned}$$

and a comparison with the known expression (2.30) for Φ_0 shows that B must have the value

$$B = \tfrac{1}{2}(3\log 2-1+\gamma-i\pi). \tag{2.41}$$

The constants A and B of formulae (2.36) and (2.37) are now determined, and it remains to calculate Φ_1, Φ_2 and Φ_3. The function Φ_1 has the simple wave-free solution

$$\Phi_1 = (R\sin\theta-1), \tag{2.42}$$

and since $\Phi_2 \sim -(1/2^{\frac{1}{2}}\pi)R^{\frac{1}{2}}\sin\frac{1}{2}\theta$ at infinity, it is proportional to the known function Φ_0; thus

$$\Phi_2 = (1/2\pi)\Phi_0. \tag{2.43}$$

It remains to calculate Φ_3, which has the limiting behaviour

$$\Phi_3 \sim \tilde{\Phi}_3 = -2^{-\frac{3}{2}}R^{\frac{3}{2}}\sin\tfrac{3}{2}\theta + \frac{3}{2^{\frac{5}{2}}\pi}((\pi-\theta)\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta\log R)R^{\frac{1}{2}}$$
$$+\frac{1}{2^{\frac{3}{2}}\pi}(3\log 2+\tfrac{3}{2}+\tfrac{1}{2}\gamma-\tfrac{1}{2}i\pi)R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta \quad (2.44)$$

for large R. It is convenient to subtract off the large wave-free potential $\tilde{\Phi}_3$ by writing

$$\Phi_3 = \tilde{\Phi}_3 + G. \quad (2.45)$$

Thus G is the harmonic function that has an edge condition (2.4) and is such that

$$\partial G(X,0)/\partial Y = 0 \quad (X<0); \quad G+\partial G/\partial Y = P(X) \quad (X>0), \quad (2.46)$$

where
$$P(X) = (3\log X - 6\log 2 + 3 - \gamma + i\pi)/2^{\frac{7}{2}}\pi X^{\frac{1}{2}}. \quad (2.47)$$

The solution for such a problem has already been given by formulae (2.22)–(2.24). In this case the transform $\Pi(s)$ has the value

$$\Pi(s) = -(3\log s + 3 + i\pi - 12\log 2 - 4\gamma)/2^{\frac{7}{2}}\pi^{\frac{3}{2}}s^{\frac{1}{2}}, \quad (2.48)$$

and the regular wave train radiated towards $x=\infty$ is given, according to (2.24) and (2.11), as $G \sim A_3\exp\{(ix-y)/\epsilon\}$, where

$$A_3 = \frac{-ie^{-\frac{1}{8}i\pi-i/\epsilon}}{8\pi^{\frac{3}{2}}}\left\{-3\int_0^\infty \frac{\log s}{s^{\frac{1}{2}}}\frac{\Lambda(s)}{1+s^2}ds + (3+i\pi-12\log 2-4\gamma)\int_0^\infty \frac{\Lambda(s)\,ds}{s^{\frac{1}{2}}(1+s^2)}\right\}.$$

Each of the integrals has the value π, as is found by integrating the functions $(s\log s-\pi)/s^{\frac{3}{2}}\Lambda(s)$ and $1/s^{\frac{1}{2}}\Lambda(s)$ round a closed contour slotted along the negative real axis, bounded by circles of small radius, ρ_1, and large radius, ρ_2. Thus

$$A_3 = \frac{-ie^{-\frac{1}{8}i\pi-i/\epsilon}}{8\pi^{\frac{1}{2}}}(-12\log 2-4\gamma+i\pi). \quad (2.49)$$

Collecting together the results (2.40), (2.28), (2.43) and (2.49), it is seen that

$$\Phi \sim A_+\exp\{(ix-y)/\epsilon\} \quad \text{as} \quad X\to\infty, \quad (2.50)$$

where
$$A_+ \sim -i\pi^{-\frac{1}{2}}e^{-i/\epsilon-\frac{1}{8}i\pi}\{\pi\epsilon^{\frac{1}{2}}+\tfrac{1}{2}\epsilon^{\frac{3}{2}}\log\epsilon+\tfrac{1}{8}\epsilon^{\frac{3}{2}}(-12\log 2-4\gamma+i\pi)\}. \quad (2.51)$$

The solution in the surface-wave region is again obtained by superimposing the outer solution (2.31) on the wave train (2.50). Thus A_+ gives the amplitude of the potential, the first two terms agreeing with earlier results due to Holford (1964) and Leppington (1970).

3. Radiation by a finite dock on a fluid of finite depth

The method described in detail in the prototype problem of § 2 can readily be extended to much more general geometries; with little modification we can deal, in principle, with the case of finite depth h and with a dock of any (two-dimensional) shape, provided that each end of the dock is horizontal for a distance of many wavelengths, and provided that the depth is many wavelengths.

In this case the surface waves have the modified length $2\pi\epsilon_1$, where

$$\epsilon = \epsilon_1\coth(h/\epsilon_1);$$

ϵ_1 differs from ϵ by an exponentially small term when $h/\epsilon \gg 1$, and the far-field amplitudes are obtained as before, with ϵ changed to ϵ_1 in the wave trains corresponding to (2.5).

For such a geometry the first outer approximation ϕ_0 is specified by setting $\phi_0 = 0$ on the free surface. The local edge behaviour of this function will be of precisely the same form (2.9) as in the prototype problem, but with a different factor I_1 in place of the constant I appearing in (2.9). The inner potential, $\Phi \sim \epsilon^{\frac{1}{2}}\Phi_0$, and the surface-wave approximation are exactly as before, with the constant I of (2.20) and (2.28) replaced by I_1; the precise geometry of the problem makes its presence felt only in setting the scale constant I_1.

As an illustration of this generalization, our attention is turned to the case in which $V(x) = 1$ is constant, and the fluid is of constant finite depth h. The function ϕ_0 vanishes at infinity and has the boundary conditions

$$\partial\phi_0(x,h)/\partial y = 0; \quad \partial\phi_0(x,0)/\partial y = 1 \quad (|x| < 1); \quad \phi_0(x,0) = 0 \quad (|x| > 1). \tag{3.1}$$

We require only the behaviour of ϕ_0 near the edge $x = 1$, where

$$\phi_0 \sim I_1 \delta^{\frac{1}{2}} \sin\tfrac{1}{2}\theta, \tag{3.2}$$

and I_1 is to be found.

The potential ϕ_0 is found by transforming the fluid region, $0 < y < h$, in the z plane ($z = x+iy$) to the interior of a rectangle in the complex-ζ plane. Defining the constant

$$k = \tanh(\pi/2h), \tag{3.3}$$

the required transformation is

$$z = \frac{h}{\pi}\left\{\log\left(\frac{k\,\mathrm{sn}\,\zeta+1}{k\,\mathrm{sn}\,\zeta-1}\right)+i\pi\right\}, \tag{3.4}$$

where sn denotes an elliptic function and the imaginary part of the logarithmic function is to lie between $-\pi$ and 0. The dock ($|x| < 1$, $y = 0$) and the bottom surface ($-\infty < x < \infty$, $y = h$) map onto the opposite sides of a rectangle, $|\xi| < K$, $\eta = 0$ and $|\xi| < K, \eta = K'$, while the free surface maps onto the other sides ($\xi = \pm K, 0 < \eta < K'$). The constants K and K' are given in terms of the elliptic integral

$$K(k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta \tag{3.5}$$

as

$$K = K(k), \quad K' = K[(1-k^2)^{\frac{1}{2}}]. \tag{3.6}$$

In terms of the new variables ξ and η ($\zeta = \xi + i\eta$), we have to find a harmonic function $\phi_0(\xi,\eta)$ satisfying the boundary conditions

$$\phi_0(\pm K,\eta) = 0, \quad 0 < \eta < K'; \quad \partial\phi_0(\xi,K')/\partial\eta = 0, \quad |\xi| < K; \tag{3.7}$$

and

$$\partial\phi_0(\xi,0)/\partial\eta = W(\xi) \equiv (2hk/\pi)\,\mathrm{cd}\,\xi, \quad |\xi| < K, \tag{3.8}$$

where $\mathrm{cd}\,\xi = (1-\mathrm{sn}^2\xi)^{\frac{1}{2}}/(1-k^2\,\mathrm{sn}^2\xi)^{\frac{1}{2}}$.

This problem is readily handled by the standard procedure of separating variables, which leads to the solution

$$\phi_0(\xi,\eta) = -\frac{2}{K}\int_0^K W(\xi')\,d\xi' \sum_0^\infty \frac{\cos\alpha_n\xi'\cos\alpha_n\xi\cosh\alpha_n(K'-\eta)}{\alpha_n\sinh\alpha_n K'}, \tag{3.9}$$

where $\alpha_n = (n+\frac{1}{2})\pi/K$. To calculate the constant I_1 in (3.2), it is convenient to examine the behaviour of ϕ_0 on the dock ($\theta = \pi$); in terms of ξ and η, ϕ_0 has the equivalent form

$$\phi_0(K-t, +0) \sim I_1(hk/\pi)^{\frac{1}{2}}t \quad \text{as} \quad t \to 0. \tag{3.10}$$

Now the exact solution (3.9) shows that

$$\phi_0(K-t, +0) \sim -\frac{2}{K}t\int_0^K W(\xi')g(\xi')\,d\xi', \tag{3.11}$$

where

$$\begin{aligned} g(\xi') &= \lim_{\eta\to 0} \sum_0^\infty (-1)^n \cos\alpha_n \xi' \frac{\cosh\alpha_n(K'-\eta)}{\sinh\alpha_n K'} \\ &= \lim_{\eta\to 0} \sum_0^\infty (-1)^n \cos\alpha_n \xi' \left\{e^{-\alpha_n\eta} + \frac{2q^{2n+1}}{1-q^{2n+1}}\right\}, \quad q = e^{-\pi K'/K}, \\ &= \tfrac{1}{2}\sec(\tfrac{1}{2}\pi\xi/K) + \frac{K}{\pi}\left\{\mathrm{dc}\,\xi - \frac{\pi}{2K}\sec\left(\frac{\pi\xi}{2K}\right)\right\} = \frac{K}{\pi}\mathrm{dc}\,\xi, \end{aligned} \tag{3.12}$$

from Abramowitz & Stegun (1965, p. 575); the device of taking the limit $\eta \to 0$ is used merely to improve the convergence of the sum for g. Thus we have, from (3.10) and (3.12),

$$I_1 = -4(hk)^{\frac{1}{2}}\pi^{-\frac{3}{2}}K(k). \tag{3.13}$$

The amplitude A_+ of waves generated towards $x = \infty$ in our radiation problem is now estimated directly from formula (2.28), with I replaced by I_1; thus

$$A_+ \sim -i\epsilon^{\frac{1}{2}}2^{\frac{3}{2}}\pi^{-1}\{h\tanh(\pi/2h)\}^{\frac{1}{2}}K\{\tanh(\pi/2h)\}e^{-\frac{1}{8}i\pi - i/\epsilon}. \tag{3.14}$$

In the limit of great depth, the elliptic integral $K \to \frac{1}{2}\pi$ as $h \to \infty$, and we recover the leading term of formula (2.51). The estimate (3.14) is uniformly valid for all $h \geqslant h_0 > 0$, but is not uniformly valid near $h = 0$ since it is essential that the wavelength is much less than depth ($\epsilon \ll h$). Our units are such that the velocity and dock half-width have the value unity. For a dock of width $2a$ and velocity V_0, we multiply the expression (3.14) by aV_0, and replace h and ϵ by h/a and ϵ/a.

4. Prototype scattering problem: the finite dock

The ideas that have been used above apply equally well to the related scattering problems, in which a regular wave progressing from $x = +\infty$ is scattered by a *fixed* dock. A convenient prototype problem is again that of a finite dock ($|x| < 1$, $y = 0$) on the surface of a fluid of infinite depth.

The incident wave is taken as

$$\phi_i = 2^{-\frac{1}{2}}\exp\{[-i(x-1)-y]/\epsilon + \tfrac{1}{8}i\pi\} \tag{4.1}$$

and the total potential $\phi(x, y; \epsilon)$ is the solution of the boundary-value problem

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = 0 \quad (y > 0), \tag{4.2}$$

with

$$\partial\phi/\partial y = 0 \quad (|x| < 1, \quad y = 0); \quad \phi + \epsilon\,\partial\phi/\partial y = 0 \quad (|x| > 1, \quad y = 0). \tag{4.3}$$

The potential is subject to the usual edge conditions (2.4) and satisfies the radiation conditions

$$\phi - \phi_i \sim 2^{-\frac{1}{2}}\tilde{R}\exp\left\{\frac{i(x-1)-y}{\epsilon} - \tfrac{1}{8}i\pi\right\}, \quad \frac{x-1}{\epsilon} \to +\infty, \tag{4.4}$$

and

$$\phi \sim 2^{-\frac{1}{2}}\tilde{T}\exp\left\{\frac{-i(x+1)-y}{\epsilon} - \tfrac{1}{8}i\pi\right\}, \quad \frac{x+1}{\epsilon} \to -\infty. \tag{4.5}$$

The reflexion and transmission constants $\tilde{R}$ and $\tilde{T}$ are unknowns of the problem, to be found approximately in the asymptotic limit $\epsilon \to 0$; these particular forms for the incident, reflected and transmitted wave trains have been chosen for ease of comparison with earlier results (Holford 1964; Leppington 1968).

Proceeding as before, we specify an outer approximation ϕ_0, valid many wavelengths from the free surface, by setting $\phi_0 = 0$ on $y = 0$, $|x| > 1$. Evidently ϕ_0 is an eigensolution of the problem and has an overall scale constant that is as yet unknown, even in order of magnitude with respect to ϵ. Thus we write

$$\phi \sim \alpha(\epsilon)\,\phi_0(\mathbf{x}), \tag{4.6}$$

with

$$\phi_0 = \mathscr{R}\left\{Ai\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} - Bi\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}}\right\}, \tag{4.7}$$

where $\alpha(\epsilon)$ and the real constants A and B are to be calculated by matching. Higher order eigensolutions are rejected: if they are included, then the matching procedure is found to require zero coefficients. In order to match with an inner expansion $\Phi(X, Y; \epsilon)$ near the edge $x = 1$, we write

$$x = 1+\epsilon X, \quad y = \epsilon Y; \quad z = 1 + Re^{i\theta} \tag{4.8}$$

(see figure 1) and expand for small ϵ to get

$$\alpha(\epsilon)\,\phi_0 = \alpha(\epsilon)2^{\frac{1}{2}}\left\{\frac{A\sin\frac{1}{2}\theta}{\epsilon^{\frac{1}{2}}R^{\frac{1}{2}}} + \epsilon^{\frac{1}{2}}R^{\frac{1}{2}}\sin\tfrac{1}{2}\theta(-\tfrac{1}{4}A + \tfrac{1}{2}B) + \dots\right\}. \tag{4.9}$$

This suggests an inner expansion of the form

$$\Phi(X, Y; \epsilon) \sim \epsilon^{-\frac{1}{2}}\alpha(\epsilon)\,\Phi_0(X, Y), \tag{4.10}$$

where

$$\Phi_0 \sim 2^{\frac{1}{2}}AR^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta \quad \text{as} \quad R \to \infty, \tag{4.11}$$

together with travelling waves.

In terms of the inner variables X and Y, given by (4.8), it is found that the harmonic function Φ_0 has to satisfy the boundary conditions

$$\partial\Phi_0(X,0)/\partial Y = 0 \quad (X < 0); \quad \Phi_0(X,0) + \partial\Phi_0(X,0)/\partial Y = 0 \quad (X > 0), \tag{4.12}$$

together with the condition (4.11).

Now this problem for Φ_0 has a non-trivial solution only if Φ_0 has the same order of magnitude as the incident wave, in order that it may possess both incoming and outgoing waves at infinity. According to the scaling (4.10), we require therefore that

$$\alpha(\epsilon) = \epsilon^{\frac{1}{2}}, \tag{4.13}$$

in which case Φ_0 is the potential for the problem of scattering the incident wave

$$\Phi_i = 2^{-\frac{1}{2}} \exp(-iX - Y + \tfrac{1}{8}i\pi) \tag{4.14}$$

by a semi-infinite rigid dock. This problem has the solution given by Holford (1964) in the form

$$\Phi_0(X, Y) = 2^{\frac{1}{2}} \cos(X - \tfrac{1}{8}\pi) e^{-Y} - \frac{1}{\pi} \int_0^\infty \frac{(t \cos tY - \sin tY)}{(1+t^2)\Lambda(t)} e^{-tX} dt \quad (X > 0), \tag{4.15}$$

with Λ defined by formula (2.25). In particular, the solution for large X is seen to be

$$\Phi_0 - \Phi_i \sim 2^{-\frac{1}{2}} \exp(iX - Y - \tfrac{1}{8}i\pi) + \pi^{-\frac{1}{2}} R^{-\frac{1}{2}} \sin \tfrac{1}{2}\theta, \tag{4.16}$$

which shows, from (4.4), that $\tilde{R} \sim 1$ to this order of accuracy. This formula also shows, by comparison with (4.11), that

$$A = 1/(2\pi)^{\frac{1}{2}}. \tag{4.17}$$

In order to derive a similar approximation $\Psi(X_1, Y_1)$ near the other edge $x = -1$, we write

$$x = -1 - \epsilon X_1, \quad y = \epsilon Y_1; \quad z = -1 - \epsilon R_1 e^{-i\theta_1} \tag{4.18}$$

(see figure 1) and expand for small ϵ to get

$$\alpha(\epsilon)\phi_0 = \epsilon^{\frac{1}{2}} 2^{\frac{1}{2}} \left\{ \frac{B \sin \frac{1}{2}\theta_1}{\epsilon^{\frac{1}{2}} R_1^{\frac{1}{2}}} + \epsilon^{\frac{1}{2}} R_1^{\frac{1}{2}} \sin \tfrac{1}{2}\theta_1 (-\tfrac{1}{4}B + \tfrac{1}{2}A) \ldots \right\}. \tag{4.19}$$

This suggests an inner expansion $\Psi \sim \Psi_0$, with $\Psi_0 \sim 2^{\frac{1}{2}} B R_1^{-\frac{1}{2}} \sin \frac{1}{2}\theta_1$, together with a purely outgoing wave. The only solution that satisfies an edge condition and radiation condition is the trivial one, $\Psi_0 \equiv 0$, so that $B = 0$, and the eigenfunction ϕ_0 can have no singularity at the edge $x = -1$. The first non-zero term near $x = -1$ is therefore given as

$$\epsilon^{\frac{1}{2}} \phi_0 \sim \tfrac{1}{2} \pi^{-\frac{1}{2}} \epsilon R_1^{\frac{1}{2}} \sin \tfrac{1}{2}\theta_1, \tag{4.20}$$

so that

$$\Psi \sim \epsilon \Psi_1. \tag{4.21}$$

The potential Ψ_1 is harmonic, and such that

$$\partial\Psi_1(X_1, 0)/\partial Y_1 = 0 \quad (X_1 < 0); \quad \Psi_1 + \partial\Psi_1/\partial Y = 0 \quad (X_1 > 0), \tag{4.22}$$

with

$$\Psi_1 \sim \tfrac{1}{2} \pi^{-\frac{1}{2}} R_1^{\frac{1}{2}} \sin \tfrac{1}{2}\theta_1 \quad \text{as} \quad R_1 \to \infty, \tag{4.23}$$

together with an outgoing wave train.

This semi-infinite dock problem is just like that (cf. equations (2.20) and (2.21)) already discussed in §2, differing only by a constant factor. In particular then,

$$\Psi \sim \epsilon \Psi_1 \sim 2^{-\frac{1}{2}} \tilde{T} \exp(iX_1 - Y_1 - \tfrac{1}{8}\pi i) \quad \text{as} \quad X_1 \to \infty, \tag{4.24}$$

where

$$\tilde{T} \sim \tfrac{1}{2}\epsilon i \quad \text{as} \quad \epsilon \to 0. \tag{4.25}$$

The reflexion coefficient $\tilde{R}$ is estimated similarly, by referring to the expression (4.9), with $A = (2\pi)^{-\frac{1}{2}}$ and $B = 0$, which implies that the inner expansion near the edge $x = 1$ has the development

$$\Phi \sim \Phi_0 + \epsilon \Phi_1, \tag{4.26}$$

with

$$\Phi_1 \sim -\tfrac{1}{4} \pi^{-\frac{1}{2}} R^{\frac{1}{2}} \sin \tfrac{1}{2}\theta \quad \text{as} \quad R \to \infty. \tag{4.27}$$

Evidently Φ_1 is proportional to Ψ_1, and is obtained by multiplying by $-\frac{1}{2}$ and changing (R_1, θ_1) to (R, θ). Thus

$$\Phi \sim 2^{-\frac{1}{2}}\tilde{R}\exp(iX - Y - \tfrac{1}{8}\pi i), \tag{4.28}$$

where

$$\tilde{R} \sim 1 - \tfrac{1}{4}i\epsilon. \tag{4.29}$$

Although Ψ_1, and similarly Φ_1, are in the first instance valid only in the inner regions that are a small fraction of a wavelength from an edge, the induced wave trains (4.24) and (4.28) are again continued into the surface-wave regions.

5. Scattering by a finite dock on a fluid of finite depth

Similar arguments may be used to obtain the corresponding reflexion and transmission constants for more general geometries. If the fluid has (variable) depth of many wavelengths and if the dock is horizontal near both ends, then the forms of the outer expansion ϕ_0 and the inner expansion Φ_0 and Ψ_1 will be similar to those described in the prototype problem of §4.

The first outer approximation $\epsilon^{\frac{1}{2}}\phi_0$, which satisfies the limiting free-surface condition $\phi_0 = 0$, is the eigensolution that has no singularity at $x = -1$ and is such that $\phi_0 = O(\delta^{-\frac{1}{2}})$ near $x = +1$. More precisely, the expansions near the two edges will be of the form

$$\phi_0 \sim C_1\delta^{-\frac{1}{2}}\sin\tfrac{1}{2}\theta + C_2\delta^{\frac{1}{2}}\sin\tfrac{1}{2}\theta \quad \text{as} \quad \delta \to 0 \tag{5.1}$$

and

$$\phi_0 \sim C_3\delta_1^{\frac{1}{2}}\sin\tfrac{1}{2}\theta_1 \quad \text{as} \quad \delta_1 \to 0, \tag{5.2}$$

near the edges $\delta = 0$ and $\delta_1 = 0$ (figure 1), where the constants C_1, C_2 and C_3 depend on the precise geometry of the problem.

Once those constants are known, the analysis proceeds exactly as before, with the two approximations Φ_1 and Ψ_1 being constant multiples of the corresponding functions of §4. It is readily found that the reflexion and transmission constants $\tilde{R}$ and $\tilde{T}$ are then given asymptotically by the general expressions

$$\tilde{R} - 1 \sim \epsilon i C_2/C_1; \quad \tilde{T} \sim \epsilon i C_3/C_1 \quad \text{as} \quad \epsilon \to 0 \tag{5.3}$$

with ϵ replaced by ϵ_1 ($\epsilon = \epsilon_1 \coth(h/\epsilon)$) in formulae (4.4) and (4.5).

It is seen again that the precise geometry of the problem influences the amplitude of the regular wave trains only in the scaling of the potential ϕ_0 near the edges.

An explicit illustration is again provided for the special case of a finite dock of width 2 units on a fluid of constant finite depth h.

The eigensolution ϕ_0, which has to vanish on the free surface $|x| > 1, y = 0$, and has vanishing derivative $\partial\phi_0/\partial y$ on the dock $|x| < 1, y = 0$, is calculated by means of the conformal transformation (3.4). In the ξ, η plane, we have to find a harmonic function within the rectangle $|\xi| < K$, $0 < \eta < K'$, with ϕ_0 vanishing on the sides $\xi = \pm K$, and $\partial\phi_0/\partial\eta$ vanishing on the other two sides. The singularity, that occurs at $(\xi, \eta) = (K, 0)$, is such that $\phi_0 = O(1/t)$, where t is the distance from $(K, 0)$. Evidently ϕ_0 corresponds to a dipole at $(K, 0)$ with its axis along the ξ axis, with the given homogeneous boundary conditions on the sides of the rectangle. By separating variables, ϕ_0 is found to have the series representation

$$\phi_0(\xi,\eta) = -(1/K)\sum_0^\infty(-1)^n\sin[\beta_n(\xi+K)]\frac{\cosh[\beta_n(K'-\eta)]}{\sinh(\beta_n K')}, \tag{5.4}$$

where $\beta_n = n\pi/2K$. In particular, letting $\eta\to 0$,

$$\begin{aligned} K\phi_0(\xi,+0) = {} & \lim_{\eta\to 0}\sum_0^\infty(-1)^n\cos(\beta_{2n+1}\xi)\frac{\cosh[\beta_{2n+1}(K'-\eta)]}{\sinh(\beta_{2n+1}K')} \\ & -\lim_{\eta\to 0}\sum_0^\infty(-1)^n\sin(\beta_{2n}\xi)\frac{\cosh[\beta_{2n}(K'-\eta)]}{\sinh(\beta_{2n}K')}. \end{aligned}$$

The first of these sums has already been dealt with (formula (3.12)) and has the value $(K/\pi)\,\mathrm{dc}\,\xi$. Manipulating the second sum in the same way, in terms of the node $q = e^{-\pi K'/K}$, it is found that

$$\begin{aligned} \pi\phi_0(\xi,0) &= \mathrm{dc}\,\xi - \lim_{\eta\to 0}\sum_0^\infty(-1)^n\sin\beta_{2n}\xi e^{-\beta_{2n}\eta} - \sum_0^\infty(-1)^n\sin\left(\frac{n\pi\xi}{K}\right)\frac{q^{2n}}{1-q^{2n}} \\ &= \mathrm{dc}\,\xi(1+\mathrm{sn}\,\xi)-\mathrm{Z}(\xi), \end{aligned} \tag{5.5}$$

from Abramowitz & Stegun (1965, pp. 577–578), where dc, sn and Z are elliptic functions.

From this exact expression for $\phi_0(\xi,0)$, it is required only to find the constants C_1, C_2 and C_3 of formulae (5.1) and (5.2). Setting θ and θ_1 equal to π relates the required constants to the value of ϕ_0 on the dock itself; equivalently, we have to examine the behaviour of $\phi_0(\xi,0)$ near the points $\xi = \pm K$. Setting $\xi = K-t$ with t small, the relation between $\delta = 1-x$ and t is found from the mapping (3.4) to be

$$t = (\pi/kh)^{\frac{1}{2}}\delta^{\frac{1}{2}}\left\{1+\delta\left(\frac{1+k^2}{24kh}\right)+\dots\right\}. \tag{5.6}$$

Similarly, if $\xi = -K+t_1$, then the small displacement t_1 is given in terms of δ_1 by the same formula (5.6) with (δ,t) replaced by (δ_1,t_1).

Now the exact solution (5.5) is found (cf. Abramowitz & Stegun 1965) to have the limiting forms

$$\pi\phi_0(K-t,0) \sim \frac{2}{t}+t\left\{\frac{5-k^2}{6}-\frac{E(k)}{K(k)}\right\} \quad \text{as} \quad t\to 0, \tag{5.7}$$

and
$$\pi\phi_0(-K-t_1,0) \sim \left\{\frac{k^2-1}{2}+\frac{E(k)}{K(k)}\right\}t_1 \quad \text{as} \quad t_1\to 0; \tag{5.8}$$

the constant E is the elliptic integral

$$E(k) = \int_0^{\frac{1}{2}\pi}(1-k^2\sin^2\theta)^{\frac{1}{2}}d\theta. \tag{5.9}$$

On substituting for t and t_1 from (5.6), and comparing with the general forms (5.1) and (5.2), the constants C_1, C_2 and C_3 are calculated. Finally, the estimates (5.3) show that

$$\tilde{R}-1 \sim \epsilon i(\pi/8kh)\{3-4E/K-k^2\} \tag{5.10}$$

and
$$\tilde{T} \sim -\tfrac{1}{2}\epsilon i\{3-4E/K-k^2\}/\{1-2E/K+k^2\}, \tag{5.11}$$

where k is given by (3.3).

6. Scattering by a T-shaped dock

Calculations similar to those of §5 lead to an asymptotic estimate for the scattering of a regular wave train (4.1) by a T-shaped dock that occupies the region $|x| < 1, y = 0$ and $x = 0, 0 < y < d$. The fluid depth is taken to be infinite.

The outer approximation $\phi \sim \epsilon^{\frac{1}{2}}\phi_0$ requires knowledge of the harmonic function ϕ_0 that vanishes on the free surface and has zero normal derivative on the dock. Now the transformation

$$u+iv = w = (z^2+d^2)^{\frac{1}{2}}, \tag{6.1}$$

with $0 < \arg w < \pi$, maps the dock onto the region $v = 0$, $|u| < (1+d^2)^{\frac{1}{2}}$, with fluid in the half-space $v > 0$. In terms of w, the solution for ϕ_0 is

$$\phi_0 = A\mathscr{R}i\left\{\frac{w+(1+d^2)^{\frac{1}{2}}}{w-(1+d^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \tag{6.2}$$

In particular, this solution implies that ϕ_0 has the edge behaviour

$$\phi_0 \sim A2^{\frac{1}{2}}(1+d^2)^{\frac{1}{4}}\left\{\frac{\sin\frac{1}{2}\theta}{\delta^{\frac{1}{2}}} - \frac{(1-d^2)}{4(1+d^2)}\delta^{\frac{1}{2}}\sin\tfrac{1}{2}\theta\right\} \quad \text{as} \quad \delta \to 0 \tag{6.3}$$

and

$$\phi_0 \sim A2^{\frac{1}{2}}(1+d^2)^{\frac{1}{4}}\left\{\frac{\delta_1^{\frac{1}{2}}\sin\frac{1}{2}\theta_1}{2(1+d^2)}\right\} \quad \text{as} \quad \delta_1 \to 0. \tag{6.4}$$

It follows at once, from formulae (5.1)–(5.3), that the reflexion and transmission constants of (4.4)–(4.5) are given asymptotically by

$$\tilde{R}-1 \sim -\epsilon i\frac{(1-d^2)}{4(1+d^2)}, \quad \tilde{T} \sim \frac{\epsilon i}{2(1+d^2)} \quad \text{as} \quad \epsilon \to 0. \tag{6.5}$$

The main point of interest here is that the transmission coefficient is only algebraically small for small ϵ; one might have expected that $\tilde{T}$ would be exponentially small, since the incident wave (4.1) is exponentially small at the bottom of the scatterer.

Further generalization

The present analysis indicates that both $\tilde{R}-1$ and $\tilde{T}$ will be of order ϵ for *any* geometry, provided that the dock is horizontal within a neighbourhood of each edge.

It seems likely that similar ideas may be used to deal with scattering objects that intersect the free surface at angles other than the value π of this work. Suppose that the front end ($x = 1$) is at angle $(p/q)\pi$ to the free surface and the back end is at angle $(p_1/q_1)\pi$ to the free surface, then calculations on the lines of those given above for the dock problem predict that

$$(\tilde{R}-1) = O(\epsilon^{q/p}), \quad \tilde{T} = O(\epsilon^{\frac{1}{2}q_1/p_1+\frac{1}{2}q/p}), \tag{6.6}$$

if $p/q, p_1/q_1 \neq \frac{1}{2}$. This prediction seems plausible if the scatterer is locally plane within a neighbourhood of each end. It is possibly correct even if the object has non-zero curvature at the ends, though may need amendment in special cases. For example, if the scatterer is nearly vertical at the intersections with the free surface, i.e. $p/q \to \frac{1}{2}$ and $p_1/q_1 \to \frac{1}{2}$, then (6.6) predicts a transmission constant

$\tilde{T} \to O(\epsilon^2)$; Ursell (1961) has shown, however, that $\tilde{T} = O(\epsilon^4)$ for the semi-circular cylinder. This is not inconsistent with the prediction $\tilde{T} = O(\epsilon^2)$, since the coefficient of the leading term suggested by (6.6) might well be zero. The general arguments that have led to the predictions (6.6) do indicate, however, that the case of vertical intersection ($p/q = \frac{1}{2}$ or $p_1/q_1 = \frac{1}{2}$) is a rather special one, and not covered by the methods described here.

Appendix. The matching principle

The matching principle that gives the connexion between the outer approximation $\phi(r;\epsilon)$ and the inner approximation $\Phi(R;\epsilon)$, with $r = \epsilon R$, is of the type proposed by Van Dyke (1964); it concerns expansions that involve powers of ϵ and integral powers of $\log \epsilon$. The variable r appears in the main text as either δ or δ_1, one for each edge; the variable θ, which occurs in potentials in the text, is a parameter common to each expansion and is suppressed here.

If the outer expansion is of the form ϕ

$$\phi(r;\epsilon) \sim \sum_i \epsilon^{m_i} \sum_{s=0}^{M_i} (\log \epsilon)^s \phi_{is}(r), \tag{A 1}$$

where M_i are *finite* integers and $m_1 < m_2 < m_3 < \ldots$, we denote by $\phi^{(m)}$ $(m \in m_i)$ the expansion (A 1) up to and including terms of order ϵ^m. Rewriting $\phi^{(m)}(r;\epsilon)$ in terms of the inner variable $R = r/\epsilon$ and expanding as $\epsilon \to 0$, with R fixed, up to terms of order ϵ^n, the result is denoted by $\phi^{(m,n)}(R;\epsilon)$.

Similarly, if the inner expansion Φ is of the form

$$\Phi(R;\epsilon) \sim \sum_j \epsilon^{n_j} \sum_{t=0}^{N_j} (\log \epsilon)^t \Phi_{jt}(R), \tag{A 2}$$

where N_j are *finite* integers and $n_1 < n_2 < n_3 < \ldots$, the function $\Phi^{(n)}$ includes all terms of the expansion (A 2) up to and including those of order ϵ^n. Rewriting $\Phi^{(n)}(R;\epsilon)$ in terms of the other variable $r = \epsilon R$ and expanding for small ϵ up to terms of order ϵ^m, the result is denoted by $\Phi^{(n,m)}(r;\epsilon)$.

The *modified Van Dyke principle* used here states simply that

$$\phi^{(m,n)} \equiv \Phi^{(n,m)}. \tag{A 3}$$

This expression is not necessarily an asymptotic representation for ϕ. The 'modification' to the Van Dyke principle arises from our stipulation that all terms up to a given order ϵ^n be included: thus all terms in a group like $\epsilon^{\frac{3}{2}} \log^2 \epsilon$, $\epsilon^{\frac{3}{2}} \log \epsilon$ and $\epsilon^{\frac{3}{2}}$ must be taken together.

REFERENCES

Abramowitz, M. & Stegun, I. A. 1965 *Handbook of Mathematical Functions*. Washington: National Bureau of Standards.

Holford, R. L. 1964 *Proc. Camb. Phil. Soc.* **60**, 957, 985.

Leppington, F. G. 1968 *Proc. Camb. Phil. Soc.* **64**, 1109.

Leppington, F. G. 1970 *J. Inst. Maths. Applics.* **6**, 319.

Ursell, F. 1961 *Proc. Camb. Phil. Soc.* **57**, 638.

Van Dyke M. 1964 *Perturbation Methods in Fluid Mechanics*. Academic.